University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 7
Solutions to Purcell problems by P. Pebler

1 *Purcell 4.8   A copper wire 1 km long is connected across a 6 volt battery. The resistivity of copper is 1.7×10^{-6} ohm cm, and the number of conduction electrons per cubic centimeter is 8×10^{22}. What is the drift velocity of the conduction electrons under these circumstances? How long does it take an electron to drift once around the circuit?*

The current will be

$$I = \frac{V}{R} = \frac{VA}{\rho L}$$

and the current density

$$J = \frac{I}{A} = \frac{V}{\rho L}.$$

The current density is the charge density times the drift speed and the charge density is ne where n is the number density of electrons.

$$v = \frac{J}{ne} = \frac{V}{\rho L n e}$$

$$v = \frac{6\ V}{(1.7 \times 10^{-6}\ ohm\,cm)(10^5\ cm)(8 \times 10^{22}\ cm^{-3})(1.6 \times 10^{-19}\ C)} = 2.8 \times 10^{-3}\ cm/s$$

$$t = \frac{10^5\ cm}{2.8 \times 10^{-3}\ cm/s} = 3.6 \times 10^7\ s \simeq 1\ year$$

2 *Purcell 4.20   A black box with three terminals a, b, and c contains nothing by three resistors and connecting wire. Measuring the resistance between pairs of terminals, we find $R_{ab} = 30$ ohm, $R_{ac} = 60$ ohm, and $R_{bc} = 70$ ohm. Show that the contents cound be either of the following.*
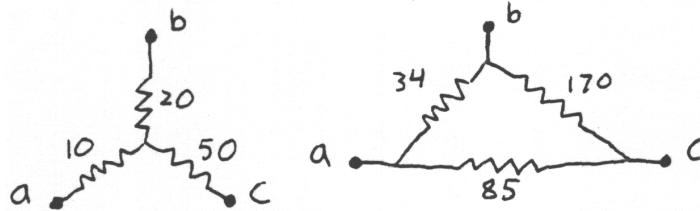
Is there any other possibility? Are the two boxes completely equivalent, or is there an external measurement that would distinguish between them?

For the first box, the resistance between any two terminals involves two of the resistors in series with the third resistor extraneous. For example, $R_{ab} = 10\ ohm + 20\ ohm = 30\ ohm$.

For the second box, the resistance between any two terminals involves one resistor in parallel with the other two in series. For example,

$$R_{ab} = \left(\frac{1}{34\ ohm} + \frac{1}{85\ ohm + 170\ ohm} \right)^{-1} = 30\ ohm.$$

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The other two are easily verified.

These are the only two ways to make these three resistances with only three resistors.

For the two arrangements to be electrically identical, they must both draw the same currents given the same input voltages V_a, V_b, and V_c. The details are somewhat messy. I'll leave it to you to verify that given the input voltages V_a, V_b, and V_c, both arrangements draw the currents

$$I_a = \frac{1}{170}(7V_a - 5V_b - 2V_c),$$

$$I_b = \frac{1}{170}(6V_b - V_c - 5V_a),$$

$$I_c = \frac{1}{170}(3V_c - V_b - 2V_a).$$

Note that $I_a + I_b + I_c = 0$ as it must.

3 *Purcell 4.25   A charged capacitor C discharges through a resistor R. Show that the total energy dissipated in the resistor agrees with the energy originally stored in the capacitor. Suppose someone objects that the capacitor is never really discharged because Q only becomes zero for $t = \infty$. How would you counter this objection?*

Assume that the capacitor initially has charge Q on it. The current as a function of time is

$$i(t) = i_o e^{-t/\tau}$$

where $\tau = RC$ and $i_o = V_o/R = Q/CR$. The power dissipated in a resistor is $P = i^2 R$, so the total energy dissipated is

$$E = \int_0^\infty P \, dt = \int_0^\infty R \, i_o^2 \exp\left(-2t/\tau\right) dt = R \, i_o^2 \frac{\tau}{2} = R \frac{Q^2}{C^2 R^2} \frac{RC}{2} = \frac{1}{2C} Q^2.$$

This was the initial energy stored in the capacitor.

We can find the time it takes for the charge left on the capacitor to be one electron. The charge as a function of time is

$$q = Q \exp\left(-t/\tau\right).$$

The time would be

$$t = \tau \ln \frac{Q}{e}.$$

Because of the ln, even for macroscopic initial charges, the time wouldn't be that large.

4 Purcell 4.26 *Two graphite rods are of equal length. One is a cylinder of radius a. The other is conical, tapering linearly from radius a at one end to radius b at the other. Show that the end-to-end resistance of the conical rod is a/b times that of the cylindrical rod.*
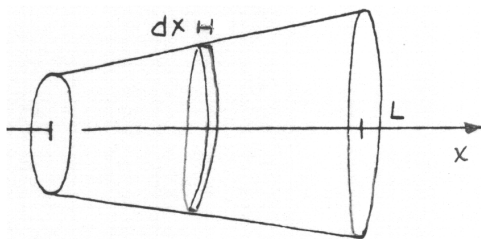
We consider the conical rod to be the series combination of little cylindrical rods of length dx. The radii of these little cylinders are

$$r(x) = a + \frac{b-a}{L}x.$$

We sum up the little resistances.

$$R = \int dR = \int_0^L \rho \frac{dx}{A} = \rho \int_0^L \frac{dx}{\pi(a + (b-a)x/L)^2} = \rho \frac{L}{\pi ab} = \frac{a}{b}\left(\rho \frac{L}{\pi a^2}\right)$$

$\rho L/\pi a^2$ is the resistance of the cylinder of radius a.

5 Purcell 4.30 *Consider two electrodes 2 mm apart in vacuum connected by a short wire. An alpha particle of charge 2e is emmitted by the left plate and travels directly towards the right plate with constant speed 10^8 cm/s and stops in this plate. Make a quantitative graph of the current in the connecting wire, plotting current against time. Do the same for an alpha particle that crosses the gap moving with the same speed but at an angle of 45°. Suppose we had a cylindrical arrangement of electrodes. Would the current pulse have the same shape?*
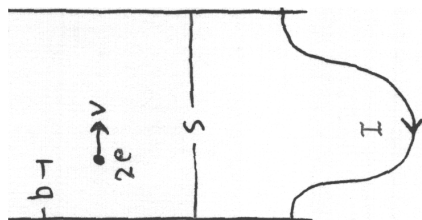
The result of problem 3.24 tells us the induced charge on the electrodes for any given position of the alpha particle. The current is just the time derivative of this charge.

$$q_1 = -2e\left(\frac{s-b}{s}\right)$$

$$I = \frac{dq_1}{dt} = \frac{2e}{s}\frac{db}{dt} = \frac{2ev}{s} = 0.48 \ esu/s$$

This pulse lasts for $(0.2 \ cm)/(10^8 \ cm/s) = 2 \ ns$. If the alpha particle travels at 45°, the normal speed and the current just decrease by $\sqrt{2}$.

For concentric cylinders, the current pulse would have a different shape. If you work out the details,

$$I \propto \frac{1}{a+vt}$$

during the time that the alpha particle is in motion.

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6 *Purcell 4.31   Suppose a cube has a resistor of resistance R_o along each edge. At each corner the leads from three resistors are soldered together. Find the equivalent resistance between two nodes that represent diagonally opposite corners of the cube. Now find the equivalent resistance between two nodes that correspond to diagonally opposite corners of one face of the cube.*
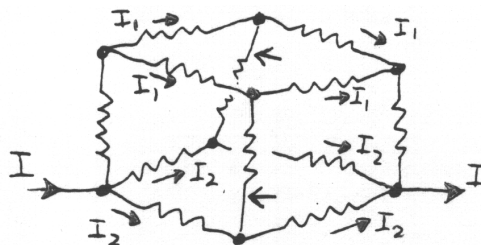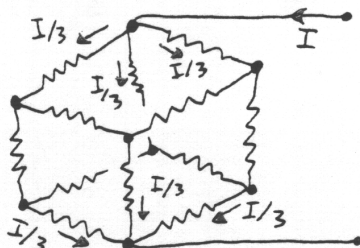
A total current I enters one node. It then has a choice of three directions to go. Because of the symmetry, each choice is identical to the others so the current must split up evenly so that $I/3$ goes through each resistor. Likewise, the current reaching the other node comes through three resistors each having current $I/3$. This leaves 6 resistors in the middle to share the current. Because each one is identical due to the symmetry, they must each have current $I/6$. To find the voltage drop between the two nodes, follow a straight path from one to the other.

$$V = \frac{I}{3}R_o + \frac{I}{6}R_o + \frac{I}{3}R_o = \frac{5}{6}RI$$

$$R_{eq} = \frac{5}{6}R_o$$

In the second situation, because of the symmetry, we notice that all the resistors on the top square carry the same magnitude of current while all the resistors on the bottom square carry the same magnitude also. This tells us that there is no current through the two resistors indicated by arrows. We can therefore ignore them, because the circuit would behave the same without them. It is then easy to combine the remaining resistors. The top and bottom squares are parallel combinations of resistors $2R_o$.

$$R = \left(\frac{1}{2R_o} + \frac{1}{2R_o}\right)^{-1} = R_o$$

This leaves us with one resistor R_o in parallel with three resistors R_o.

$$R_{eq} = \left(\frac{1}{R_o} + \frac{1}{3R_o}\right)^{-1} = \frac{3}{4}R_o$$

7 *Purcell 4.32* *Find the input resistance (between terminals A and B) of the following infinite series.*
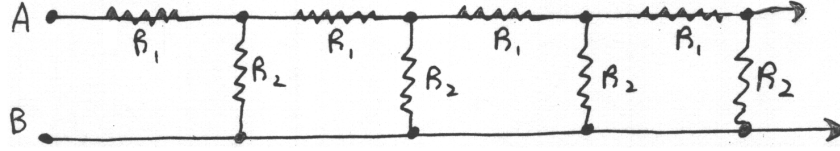
Show that, if voltage V_o is applied at the input to such a chain, the voltage at successive nodes decreases in a geometric series. What ratio is required for the resistors to make the ladder an attenuator that halves the voltage at every step? Can you suggest a way to terminate the ladder after a few sections without introducing any error in its attenuation?
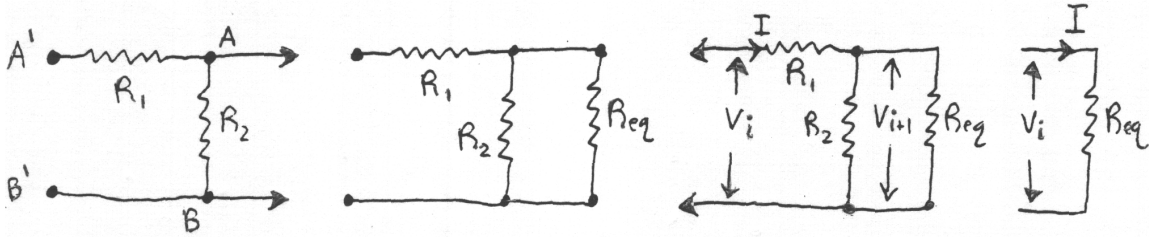
If we put another "link" on the left of this infinite chain, we get exactly the same configuration. If this infinite chain has equivalent resistance R_{eq}, the new chain with the extra link can be described by the middle circuit. We can calculate the equivalent resistance of this circuit by considering R_{eq} and R_2 in parallel, in series with R_1. But since this circuit is the same as the original, this equivalent resistance is again R_{eq}.

$$R_{eq} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_{eq}} \right)^{-1}$$

This leads to the equation

$$R_{eq}^2 - R_1 R_{eq} - R_1 R_2 = 0,$$

with positive solution

$$R_{eq} = \frac{R_1 + \sqrt{R_1^2 + 4R_1 R_2}}{2}.$$

Now consider an arbitrary link with voltage difference V_i between top and bottom. To find the voltage V_{i+1}, we can replace the rest of the series with the equivalent resistor. We could get V_{i+1} if we knew the current through R_1. For this purpose we can replace the ith link also with R_{eq}. The current through this equivalent circuit will also be the current through R_1. This current is just V_i/R_{eq}.

$$V_{i+1} = V_i - R_1 \frac{V_i}{R_{eq}} = V_i \frac{R_{eq} - R_1}{R_{eq}}$$

If we wish to halve the voltage each step,

$$\frac{R_1}{R_{eq}} = \frac{1}{2},$$

$$4R_1 = R_1 + \sqrt{R_1^2 + 4R_1 R_2},$$

$$R_2 = 2R_1.$$

If we wish to terminate the ladder without changing this property, we just replace the rest of the chain at any point with a resistor with resistance R_{eq}.

8 *Taylor & Wheeler 19* *(a.) Two events P and Q have a spacelike separation. Show that an inertial frame can be found in which the two events occur at the same time. In this frame, find the distance between the two events (this is called the proper distance). (Hint: one method of proof is to assume that such an inertial frame exists and then use the Lorentz transformation equations to show that the velocity βc of this inertial frame, relative to the frame in which the events were initially described, is such that $\beta < 1$, thus justifying the assumption made.) (b.) Two events P and R have a timelike separation. Show that an inertial frame can be found in which the two events occur at the same place. In this frame, find the time interval between the two events (this is called the proper time).*

Denote the frame in which P and Q were initially described by $\mathcal{S}$, and the frame in which we wish them to be simultaneous by $\mathcal{S}'$. As usual, the origins of these frames coincide at $t = t' = 0$. In $\mathcal{S}$, if P and Q have a spacelike separation, their spatial separation must be nonzero. Orient the x and x' axes along the direction of this separation, so that $y_Q = y_P$ and $z_Q = z_P$ but $x_Q \neq x_P$. Applying the Lorentz transformation between $\mathcal{S}'$ and $\mathcal{S}$,

$$ct'_P = \gamma ct_P - \gamma \beta x_P$$

$$ct'_Q = \gamma ct_Q - \gamma \beta x_Q$$

$$ct'_Q - ct'_P = \gamma(ct_Q - ct_P) - \gamma\beta(x_Q - x_P)$$

We wish the left-hand side to be zero. If it is, then

$$\gamma(ct_Q - ct_P) = \gamma\beta(x_Q - x_P)$$

$$\frac{(ct_Q - ct_P)}{(x_Q - x_P)} = \beta$$

Now, we are told that the separation between events Q and P is spacelike:

$$c^2(t_Q - t_P)^2 - (x_Q - x_P)^2 < 0$$

This guarantees that $|\beta| < 1$. It is straightforward to calculate the spatial separation of the two events in $\mathcal{S}'$ by using the inverse Lorentz transformation:

$$x_P = \gamma x'_P + \gamma\beta ct'_P$$

$$x_Q = \gamma x'_Q + \gamma\beta ct'_Q$$

$$x_Q - x_P = \gamma(x'_Q - x'_P) + \gamma\beta(ct'_Q - ct'_P)$$

We have chosen β so that the two events are simultaneous in $\mathcal{S}'$; this forces the last term to vanish. Substituting the value that we found for β,

$$x_Q - x_P = \gamma(x'_Q - x'_P)$$

$$x'_Q - x'_P = \sqrt{1 - \beta^2}\,(x_Q - x_P)$$

$$x'_Q - x'_P = \sqrt{1 - \left(\frac{ct_Q - ct_P}{x_Q - x_P}\right)^2}\,(x_Q - x_P)$$

$$x'_Q - x'_P = \pm\sqrt{(x_Q - x_P)^2 - (ct_Q - ct_P)^2}$$

where the sign is chosen so that $x'_Q - x'_P$ has the same sign as $x_Q - x_P$. This *proper distance* is the smallest distance between the two events that can be reached in any reference frame.

Similarly, when the separation between events P and R is timelike,

$$x'_R - x'_P = \gamma(x_R - x_P) - \gamma\beta(ct_R - ct_P)$$

$$\gamma(x_R - x_P) = \gamma\beta(ct_R - ct_P)$$

$$\frac{(x_R - x_P)}{(ct_R - ct_P)} = \beta$$

$$ct_R - ct_P = \gamma(ct'_R - ct'_P) + \gamma\beta(x'_R - x'_P)$$

$$ct'_R - ct'_P = \sqrt{1 - \beta^2}\,(ct_R - ct_P)$$

$$ct'_R - ct'_P = \sqrt{1 - \left(\frac{x_R - x_P}{ct_R - ct_P}\right)^2}\,(ct_R - ct_P)$$

$$ct'_R - ct'_P = \pm\sqrt{(ct_R - ct_P)^2 - (x_R - x_P)^2}$$

where the sign is chosen so that $ct'_R - ct'_P$ has the same sign as $ct_R - ct_P$. This *proper time* is the smallest time interval between the two events that can be reached in any reference frame.